Berry's phase in a one-dimensional quantum many-body system

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We study an interacting one-dimensional quantum lattice gas of massive fermions on a ring with L lattice sites. The ring is threaded by a magnetic flux corresponding to a twist in boundary conditions. We compute the periodicity of the ground state under an adiabatically increasing flux and the associated Berry's phase occurring in this process. The model has a second-order phase transition line which coincides with a line where the Berry phase changes nonanalytically.

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We study the quantum Hamiltonian

$$H=-\frac{1}{2\xi\cosh\lambda}\sum_{j=1}^{L/2}[e^{\lambda}u_{2j}+e^{-\lambda}u_{2j-1}+2\mu(\sigma_{2j}^z-\sigma_{2j-1}^z)] \quad (1)$$

acting on a ring of L sites (L even) where

$$u_j=\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\Delta(\sigma_j^z\sigma_{j+1}^z-1) . \quad (2)$$

Here $\sigma_j^{x,y,z}$ are the Pauli matrices acting on site j. We consider twisted boundary conditions where

$$\sigma_{L+1}^x\pm i\sigma_{L+1}^y=e^{\pm i\Phi}(\sigma_1^x\pm i\sigma_1^y) \quad (3)$$

and $\sigma_{L+1}^z=\sigma_1^z$.

For $\lambda=\mu=0$ this is the Hamiltonian of the anisotropic Heisenberg antiferromagnet which in the range $-1<\Delta<1$ is known to describe a lattice gas of massless fermions with a four-fermion coupling of strength Δ. In this particular case the model is exactly soluble and its eigenvalues and eigenvectors can be found by means of the Bethe ansatz [1,2]. The energy gaps vanish proportional to $1/L$ and H has the finite-size scaling spectrum of a conformally invariant Gauss model (Coulomb gas) with central charge $c=1$. Assigning to the fermions a charge e, the boundary twist Φ corresponds to a magnetic flux of strength $\hbar\tilde{c}\Phi/e$ threading the ring [3]. (We shall use units $\hbar=\tilde{c}=e=1$.) Obviously, the eigenvalues of H satisfy $E(\Phi)=E(\Phi+2\pi)$; the ground-state wave function, however, was shown to have a periodicity of 4π [4]. Associated to an adiabatic variation of the flux from $\Phi=0$ to $\Phi=4\pi$ is the occurrence of a Berry phase γ [5–7] in the ground state. Using the Bethe ansatz, Korepin and Wu computed Berry's phase and found $\gamma=\pi$ for $0<\Delta<1$, independent of the size L of the system [8].

With λ or μ nonzero, the model is not critical and in general not exactly soluble. Only in the noninteracting case $\Delta=0$ the model remains soluble and we find a finite energy gap $M=\sqrt{\sinh^2\lambda+\mu^2}/\cosh\lambda$ (see below). For $\Delta\neq0$ one obtains, in the scaling limit $M=m/L\to0$ with $m=$const, a system of interacting massive fermions for which very little is known. In this Brief Report we compute the periodicity of the ground state and the behavior of Berry's phase if the system is driven away from the second-order phase-transition point $\lambda=\mu=0$ by introducing the mass terms λ and μ. For our purposes the normalization ξ is irrelevant and we set $\xi=1$.

First we discuss some of the symmetries of the model. The Hamiltonian H commutes with

$$S^z=\tfrac{1}{2}\sum_{j=1}^{L}\sigma_j^z \quad (4)$$

and with the generalized two-step translation operator

$$T=[D(\Phi)]^2=(e^{i\Phi\sigma_1^z/2}D)^2 , \quad (5)$$

where the single-step translation operator D is defined by $D\sigma_jD^{-1}=\sigma_{j-1}$ for any local operator σ_j with j defined as mod$L$. The Hamiltonians $H(\lambda,\mu,\Phi)$ and $H(\lambda,-\mu,-\Phi)=PH(\lambda,\mu,\Phi)P^{-1}$ are related through the parity transformation P acting on local operators as $P\sigma_jP^{-1}=\sigma_{L+1-j}$. On the other hand, the operator $D(\Phi)$ transforms $H(\lambda,\mu,\Phi)$ into $H(-\lambda,-\mu,\Phi)=D(\Phi)H(\lambda,\mu,\Phi)D^{-1}(\Phi)$. From these relations and the hermiticity of H we conclude that its spectrum depends only on the absolute values of λ and μ. Finally, for $\mu=0$, the quantum system defined by (1) is invariant under time reversal. The antiunitary operator KP where K transforms an operator or state vector into its complex conjugate leaves $H(\lambda,0,\Phi)$ invariant. This shows that for $\mu=0$ the Berry phase must be either $\gamma=0$ or $\gamma=\pi$. The question that remains in this special case is which of the two possible values γ takes and whether it depends on the parameters α, Δ, and the length L of the chain.

The physical meaning of the parameters λ, μ, and Δ as mass terms and interaction strength, respectively, is more transparent after a Jordan-Wigner transformation: We introduce a two-component spinor by defining as its components the creation (annihilation) operators $b_k^\dagger$ and $c_k^\dagger$ (b_k and c_k) as

$$\begin{aligned}
b_k&=(-1)^k s_{2k}^+\prod_{l=1}^{2k-1}\sigma_l^z ,\\
b_k^\dagger&=(-1)^k\left[\prod_{l=1}^{2k-1}\sigma_l^z\right]s_{2k}^- ,\\
c_k&=i(-1)^k s_{2k-1}^+\prod_{l=1}^{2k-2}\sigma_l^z ,\\
c_k^\dagger&=-i(-1)^k\left[\prod_{l=1}^{2k-2}\sigma_l^z\right]s_{2k-1}^- .
\end{aligned} \quad (6)$$

They satisfy the anticommutation relations $\{b_k,b_l^\dagger\}=\{c_k,c_l^\dagger\}=\delta_{k,l}$. All other anticommutators vanish. The number operator

$$\hat{N}=\sum_{k=1}^{L/2}(b_k^\dagger b_k+c_k^\dagger c_k) \quad (7)$$

commutes with the Hamiltonian, which expressed in terms of the new operators reads

$$H=-\frac{1}{2\cosh\lambda}\sum_{k=1}^{L/2}\{ie^{\lambda}[b_k^\dagger(c_{k+1}-c_k)+b_k(c_{k+1}^\dagger-c_k^\dagger)]+2i\sinh\lambda(b_k^\dagger c_k-c_k^\dagger b_k)+2\mu(c_k^\dagger c_k-b_k^\dagger b_k)$$
$$+4\Delta b_k^\dagger b_k(e^{-\lambda}c_k^\dagger c_k+e^{\lambda}c_{k+1}^\dagger c_{k+1})\}+2\Delta\hat{N}\ . \tag{8}$$

The boundary conditions now read

$$c_{L/2+1}=c_1e^{i(\pi L/2-\pi\hat{N}+\Phi)}\ ,$$
$$c_{L/2+1}^\dagger=e^{-i(\pi L/2-\pi\hat{N}+\Phi)}c_1^\dagger\ . \tag{9}$$

Following [6] (see also the discussion in [8]), Berry's phase γ of a time-dependent eigenstate $|\psi(t)\rangle$ of H under a periodic adiabatic change of Φ at time $t=0$ to $\Phi+\Phi_0$ at time T is given by

$$\gamma=\mathrm{Re}\left[i\int_0^T dt\left\langle\psi(t)\left|\frac{\partial}{\partial t}\right|\psi(t)\right\rangle\right]$$
$$=\mathrm{Re}\left[i\int_0^{\Phi_0}d\Phi\left\langle\psi(\Phi)\left|\frac{\partial}{\partial\Phi}\right|\psi(\Phi)\right\rangle\right]\ , \tag{10}$$

where we assume the state vector to be normalized to 1,

$$\langle\psi(\Phi)|\psi(\Phi)\rangle=1\ , \tag{11}$$

and periodic with period Φ_0, i.e.,

$$|\psi(\Phi)\rangle=|\psi(\Phi+\Phi_0)\rangle\ . \tag{12}$$

In what follows we compute Φ_0 and γ for the ground state of H.

First we study the noninteracting system with $\Delta=0$. In this case H is quadratic in the fields and the model can be solved by standard techniques. We consider a sector of fixed quantum number N (7) and perform a discrete Fourier transformation by introducing the operators

$$\beta_n=\left[\frac{2}{L}\right]^{1/2}\sum_{k=1}^{L/2}e^{-ip_nk}b_k\ ,$$
$$\beta_n^\dagger=\left[\frac{2}{L}\right]^{1/2}\sum_{k=1}^{L/2}e^{ip_nk}b_k^\dagger\ ,$$
$$\gamma_n=\left[\frac{2}{L}\right]^{1/2}\sum_{k=1}^{L/2}e^{-ip_nk}c_k\ ,$$
$$\gamma_n^\dagger=\left[\frac{2}{L}\right]^{1/2}\sum_{k=1}^{L/2}e^{ip_nk}c_k^\dagger\ , \tag{13}$$

with

$$p_n=\frac{4\pi n+2\Phi+(L-2N)\pi}{2L}\ . \tag{14}$$

From this we obtain

$$H=-\frac{1}{\cosh\lambda}\sum_{n=1}^{L/2}\{i\cos(p_n+i\lambda)e^{-ip_n}\gamma_n^\dagger\beta_n$$
$$-i\cos(p_n-i\lambda)e^{ip_n}\beta_n^\dagger\gamma_n$$
$$+\mu(\gamma_n^\dagger\gamma_n-\beta_n^\dagger\beta_n)\}\ . \tag{15}$$

This Hamiltonian is diagonalized by a transformation of the fields β_n and γ_n. Defining the operators A_n and $\overline{A}_n$ as

$$A_n=\frac{\beta_n+r_n\gamma_n}{\sqrt{1+|r_n|^2}}\ , \tag{16}$$

$$\overline{A}_n=\frac{\beta_n+\overline{r}_n\gamma_n}{\sqrt{1+|\overline{r}_n|^2}}\ , \tag{17}$$

and the constants r_n and $\overline{r}_n$ by

$$r_n=\frac{-ie^{ip_n}}{\cos(p_n+i\lambda)}(\mu+\epsilon_n)\ , \tag{18}$$

$$\overline{r}_n=\frac{-ie^{ip_n}}{\cos(p_n+i\lambda)}(\mu-\epsilon_n)\ , \tag{19}$$

one obtains

$$H=-\sum_{n=1}^{L/2}\epsilon_n(A_n^\dagger A_n-\overline{A}_n^\dagger\overline{A}_n)\ , \tag{20}$$

with the excitation energies

$$\epsilon_n=\frac{\sqrt{\cos^2p_n+\sinh^2\lambda+\mu^2}}{\cosh\lambda}\ . \tag{21}$$

The fields satisfy

$$A_n=A_{n+L/2}\ ,\quad \overline{A}_n=\overline{A}_{n+L/2}\ , \tag{22}$$

and the anticommutation relations $\{A_n,A_m^\dagger\}=\{\overline{A}_n,\overline{A}_m^\dagger\}=\delta_{n,m}$. All other anticommutators vanish. The number operator (7) expressed in terms of the transformed fields reads $\hat{N}=\sum_{n=1}^{L/2}(A_n^\dagger A_n+\overline{A}_n^\dagger\overline{A}_n)$. Note that one has

$$A_n(\Phi+2\pi)=A_{n+1}(\Phi)\ ,$$
$$\overline{A}_n(\Phi+2\pi)=\overline{A}_{n+1}(\Phi)\ . \tag{23}$$

The eigenstates of H with eigenvalue $E=E(\Phi)=(-\sum_{j=1}^{k}\epsilon_{n_j}+\sum_{j=1}^{l}\epsilon_{m_j})$ are of the form

$$|n_1,\ldots,n_k,m_1,\ldots,m_l\rangle=A_{n_1}^\dagger\cdots A_{n_k}^\dagger\overline{A}_{m_1}^\dagger\cdots\overline{A}_{m_l}^\dagger|0\rangle, \tag{24}$$

where the pseudovacuum state $|0\rangle$ is the (single) eigenstate of H with quantum number $N=0$. In the spin representation this is the state with all spins up. The eigenvalues (24) with eigenvalue $N=k+l$ of the number operator $\hat{N}$ and their Hermitian conjugates form an orthonormal basis.

The ground state of the system with periodic boundary conditions, i.e., $\Phi=0$, with energy $E_0(\Phi=0) = -\sum_{n=1}^{L/2}\epsilon_n(0)$ is the state

$$|v(0)\rangle = \prod_{n=1}^{L/2} A_n^\dagger(0)|0\rangle \ . \tag{25}$$

From (22, (23), and the anticommutation relations we find

$$\begin{aligned}|v(2\pi)\rangle &= \prod_{n=2}^{L/2+1} A_n^\dagger(0)|0\rangle \\ &=(-1)^{L/2}|v(0)\rangle \ .\end{aligned} \tag{26}$$

Furthermore, by inspecting the excitation spectrum as given by (20) and (21) we find that for $\lambda\neq 0$ or $\mu\neq 0$ there is no level crossing of the ground state as Φ is varied.

Now we are in a position to compute the Berry phase of the ground state. From (26) we conclude that the periodicity of the ground state is $\Phi_0=2\pi$. In order to eliminate the L-dependent phase we redefine the ground state as $|\tilde{v}(\Phi)\rangle = e^{i\Phi L/4}|v(\Phi)\rangle$. This state satisfies the conditions (11) and (12) and we can use Eq. (10) in order to compute the Berry phase γ.

From (25) and (26) we obtain

$$\begin{aligned}\left\langle \tilde{v} \left| \frac{\partial}{\partial\Phi} \right| \tilde{v} \right\rangle &= -i\frac{L}{4} + \left\langle v \left| \frac{\partial}{\partial\Phi} \right| v \right\rangle \\ &= -i\frac{L}{4} + \sum_{n=1}^{L/2} \left\langle 0 \left| A_n \left[\frac{\partial}{\partial\Phi} A_n^\dagger \right] \right| 0 \right\rangle \\ &= -i\frac{L}{4} + \sum_{n=1}^{L/2} f\left[\frac{2\pi n+\Phi}{L}\right]\end{aligned} \tag{27}$$

and therefore

$$\begin{aligned}\gamma &= \mathrm{Re}\left[i\int_0^{2\pi} d\Phi \left\langle \tilde{v} \left| \frac{\partial}{\partial\Phi} \right| \tilde{v} \right\rangle \right] \\ &= \frac{\pi}{2}L + \mathrm{Re}\left[i\sum_{n=1}^{L/2} \int_0^{2\pi} d\Phi\, f\left[\frac{2\pi n+\Phi}{L}\right]\right] \\ &= \frac{\pi}{2}L + \mathrm{Re}\left[iL\int_0^{\pi} dx\, f(x)\right]\end{aligned} \tag{28}$$

with $x=\Phi/L$ and $f(x)=\langle 0|A_0(\partial/\partial x)A_0^\dagger|0\rangle/L$. Here we have used the periodicity $f(x)=f(x+\pi)$, which is a consequence of the periodicity relations (22).

It remains to compute $f(x)$. After some algebra we find

$$f(x)=\frac{i}{2L}\left[L+2-\left[1+\frac{\mu}{\epsilon(x)}\right]\left[1+\frac{\cosh\lambda\sinh\lambda}{\epsilon^2(x)-\mu^2}\right]\right] , \tag{29}$$

with $\epsilon(x)=\sqrt{\cos^2 x+\sinh^2\lambda+\mu^2}$. Thus we obtain Berry's phase

$$\gamma=\pi-\tfrac{1}{2}\int_0^{\pi}dx\left[1+\frac{\mu}{\epsilon(x)}\right]\left[1+\frac{\cosh\lambda\sinh\lambda}{\epsilon^2(x)-\mu^2}\right] , \tag{30}$$

which can be expressed in terms of elliptic integrals [9] as

$$\gamma=\pi\Theta(\lambda)+\sqrt{k}\left[\mu F\left[\frac{\pi}{2},k\right]-\mu\tanh\lambda\,\Pi\left[\frac{\pi}{2},n,k\right]\right] , \tag{31}$$

with

$$k^{-1}=\cosh^2\lambda+\mu^2 , \tag{32}$$

$$n^{-1}=\mu^2 \ . \tag{33}$$

The step function $\Theta(x)$ is defined as $\Theta(x)=0$ for $x<0$, $\Theta(0)=\frac{1}{2}$, and $\Theta(x)=1$ for $x>0$. Note that this result is not valid for $\lambda=\mu=0$ in which case one finds $\gamma=\pi$. Thus for $\mu=0$ we obtain

$$\gamma=\begin{cases}\pi, & \lambda\geq 0\\ 0, & \lambda<0 \ .\end{cases} \tag{34}$$

Note that for $\lambda=0$ and μ small but nonzero (31) yields

$$\gamma=\frac{\pi}{2}+\mu\ln\frac{|\mu|}{4}+\cdots \ . \tag{35}$$

Now we consider the interacting case $0<\Delta<1$. We restrict our discussion to the time-reversal-invariant case $\mu=0$. As quoted above, for $\lambda=0$ one finds $\gamma=\pi$ [8]. In the limit $\lambda\rightarrow\pm\infty$ the Hamiltonian decouples into a sum of commuting pieces u_{2j} (or u_{2j-1}, respectively) and it is easy to see that the periodicity of the ground state is $\Phi_0=2\pi$ and $\gamma=\pi$ for $\alpha\rightarrow+\infty$ while $\gamma=0$ for $\alpha\rightarrow-\infty$. Both results are independent of the interaction strength Δ and the length L of the chain and for reasons of continuity we conclude that $\gamma=0$ for $\lambda<\lambda_0$ and $\gamma=\pi$ for $\lambda\geq\lambda_0$ for some $\lambda_0<0$. In order to determine the critical value λ_0 we express the ground state $|v_\lambda(\Phi)\rangle$ of $H(\lambda,0,\Phi)$ in terms of the eigenstates of $H(0,0,\Phi)$ and study the symmetries of the coefficients in this expansion. The eigenstates $|\Psi_n(\Phi)\rangle$ of $H(0,0,\Phi)$ are also eigenstates of the one-step translation operator $D(\Phi)$ (5) with eigenvalue $\exp(ik_n)$ where $k_n=2\pi n/L$. Choosing $\lambda\neq 0$ mixes only states which differ in their total momentum by 0 or π from the momentum 0 of the ground state $|v_0(\Phi)\rangle$ of $H(0,0,\Phi)$, and we can write

$$\begin{aligned}|v_\lambda(\Phi)\rangle = \sum_n [&f_{2n}(\lambda,\Phi)|\Psi_{2n}(\Phi)\rangle \\ &+f_{2n-1}(\lambda,\Phi)|\Psi_{2n-1}(\Phi)\rangle] ,\end{aligned} \tag{36}$$

where states with an even index have momentum $k=0$ and states with an odd index have momentum $k=\pi$. We also know that $|v_{-\lambda}(\Phi)\rangle = D(\Phi)|v_\lambda(\Phi)\rangle$ and therefore

$$\begin{aligned}|v_{-\lambda}(\Phi)\rangle = \sum_n [&f_{2n}(\lambda,\Phi)|\Psi_{2n}(\Phi)\rangle \\ &-f_{2n-1}(\lambda,\Phi)|\Psi_{2n-1}(\Phi)\rangle] \ .\end{aligned} \tag{37}$$

Inserting these expressions into (10) one finds that if γ does not depend on λ for $\lambda>0$ (as we assume because of continuity) then it must be independent of λ also for $\lambda<0$ unless there are some strange cancellations. Thus we find $\lambda_0=0$ and the Berry phase $\gamma=\gamma(\lambda,\Delta)$ for the interacting system $0\leq\Delta<1$ is given by the same expression (34) as for the noninteracting model with $\Delta=0$.

To summarize, we have studied the Hamiltonian (1) of an interacting one-dimensional fermionic system on a ring. We computed the Berry phase γ of its ground state associated to an adiabatic change of a magnetic flux threading the ring as a function of the mass terms λ and μ. For $\mu=0$ the system is invariant under time reversal. In this case γ switches from π to 0 as λ becomes negative (34). We have argued that this discontinuity at $\lambda=0$ does not depend on the interaction strength Δ (in the range $0 \leq \Delta < 1$) and on the size L of the system. It is an interesting coincidence that at $\lambda=0$ the system has a second-order phase transition to a conformally invariant massless phase. The spectra for positive and negative values are identical, but not the Berry phase. One may speculate that such a relation between critical and noncritical regimes holds also for other systems. It would be interesting to study this observation, which from a physical point of view is purely accidental, more systematically in other integrable models.

For $\mu \neq 0$ one breaks time-reversal symmetry and the system has a Berry phase continuously depending on μ. For the noninteracting system γ is given by (31). It does not depend on L. It is remarkable that for $\lambda=0$ and μ small one finds $\gamma \approx \pi/2$ instead of $\gamma=\pi$ for $\mu=0$. This a consequence of a level splitting between states which are degenerate for $\mu=0$.

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